# Distance Computation Between Non-Convex Polyhedra

Christian Lennerz

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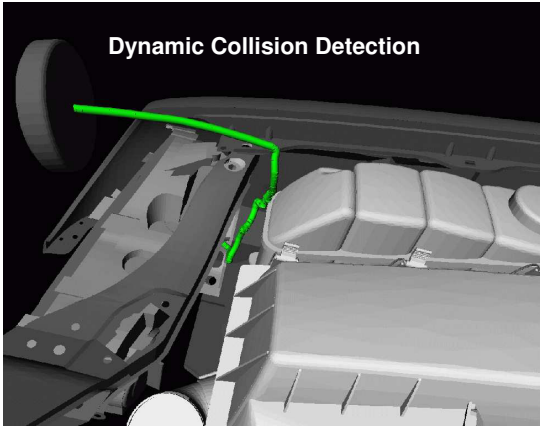
# Applications



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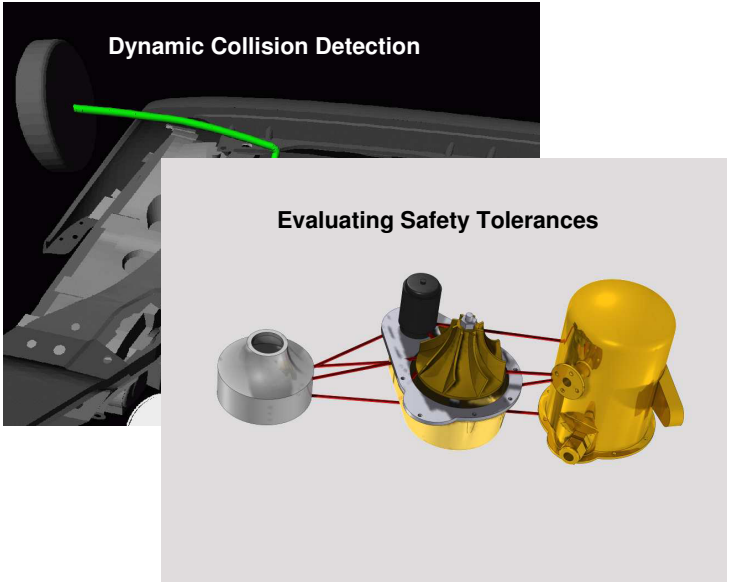

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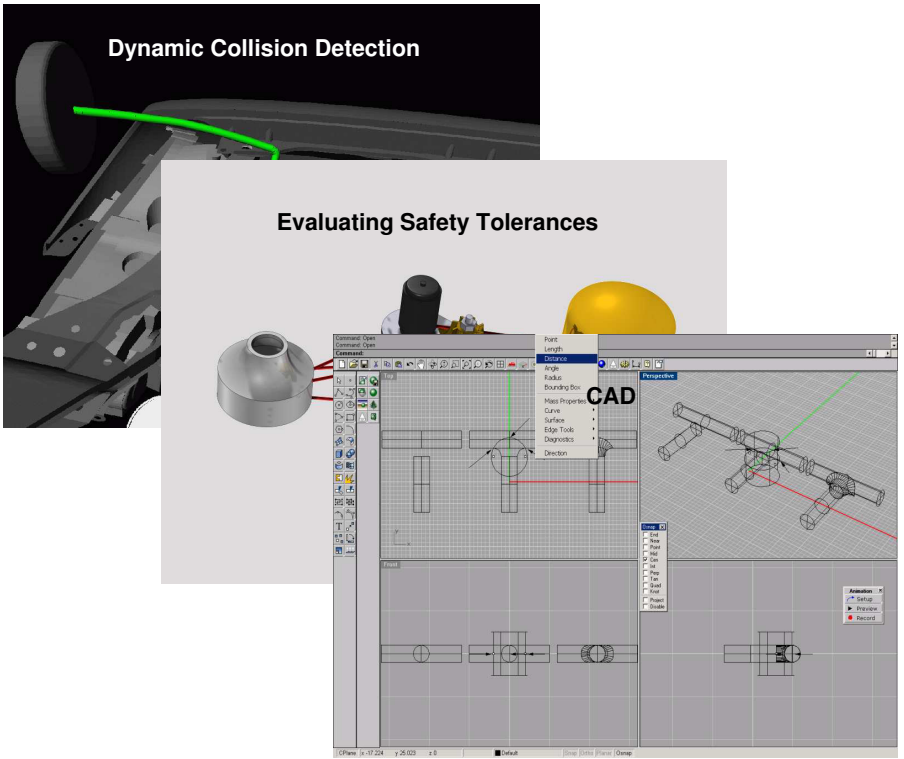

# The Distance Computation Problem

**Definition 1 (Distance Computation Problem)**
*Given two polyhedra $\mathbf{P}_1$, $\mathbf{P}_2$. The distance computation problem is to determine the global minimum of the distance function $\delta$ between the respective point sets, together with a pair of witness points i.e.*

*(i) the value $\delta^* := \delta(\mathbf{P}_1, \mathbf{P}_2)$,*

*(ii) a pair of points $(\mathbf{p}, \mathbf{q})$, s.t. $\delta^* = \delta(\mathbf{p}, \mathbf{q})$,*

*where $\delta$ denotes the (quadratic)* EUCLIDEAN *distance function between two points or set of points, respectively.*

# Reduction to Basic Proximity Tests

**Observation 1**
*Given two polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$ in boundary representation. Let $\mathcal{F}_i$, $\mathcal{E}_i$ and $\mathcal{V}_i$, $i = 1, 2$, denote the respective sets of faces, edges and vertices.*
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$$\begin{aligned}\delta(\mathcal{P}_1, \mathcal{P}_2) &= \delta(\partial\mathcal{P}_1, \partial\mathcal{P}_2) \\ &= \min\left\{\delta(f_1, f_2) \,\middle|\, f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_1\right\}\end{aligned}$$

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&= \begin{cases} \min\left\{\delta(\mathcal{F}_1, \mathcal{V}_2)\,, \delta(\mathcal{V}_1, \mathcal{F}_2),\ \delta(\mathcal{E}_1, \mathcal{E}_2)\right\} & \text{if } \mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset\,; \\ 0 & \text{else}\,. \end{cases}
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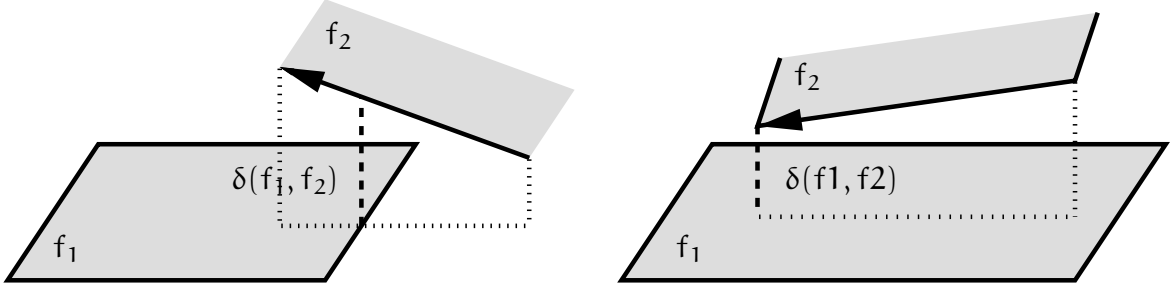

# A Simple Algorithm

DISTANCE($\mathcal{P}_1, \mathcal{P}_2$)
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# Edge-Edge-Distance (I)

Let $g_1$ and $g_2$ denote the straight lines, on which the edges are embedded:

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The quadratic distance between $g_1$ and $g_2$ is given by

$$\delta(g_1, g_2) = \left( \lambda_1 \boldsymbol{\nu}_1 - \lambda_2 \boldsymbol{\nu}_2 - \boldsymbol{\nu}_{12} \right)^2 \,,$$

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To minimize the distance function, we consider its partial derivatives with respect to $\lambda_1$ and $\lambda_2$:

$$\frac{\partial \delta(\lambda_1, \lambda_2)}{\partial \lambda_1} = 2 \left(\lambda_1 \boldsymbol{v}_1 - \lambda_2 \boldsymbol{v}_2 - \boldsymbol{v}_{12}\right)^T \boldsymbol{v}_1 = 0 \;\Leftrightarrow\; \lambda_1 \boldsymbol{v}_1^2 - \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \boldsymbol{v}_{12} \,,$$

$$\frac{\partial \delta(\lambda_1, \lambda_2)}{\partial \lambda_2} = -2 \left(\lambda_1 \boldsymbol{v}_1 - \lambda_2 \boldsymbol{v}_2 - \boldsymbol{v}_{12}\right)^T \boldsymbol{v}_2 = 0 \;\Leftrightarrow\; \lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 - \lambda_2 \boldsymbol{v}_2^2 = \boldsymbol{v}_2^T \boldsymbol{v}_{12} \,.$$

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$$\lambda_2^*(\lambda_1) = \frac{\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 - \boldsymbol{v}_2^T \boldsymbol{v}_{12}}{\boldsymbol{v}_2^2} = \frac{\lambda_1 V_{12} - W_2}{V_2}\,, \tag{2}$$

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Eliminating one variable yields $\lambda_i^*$, $i = 1, 2$, explicitly:

$$\lambda_1^* = \frac{W_1 V_2 - W_2 V_{12}}{V_1 V_2 - V_{12}^2} \qquad \lambda_2^* = \frac{W_1 V_{12} - W_2 V_1}{V_1 V_2 - V_{12}^2}\,, \tag{3}$$

if $V_1 V_2 - V_{12}^2 = \boldsymbol{v}_1^2 \boldsymbol{v}_2^2 - (\boldsymbol{v}_1^T \boldsymbol{v}_2)^2 = (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2 \neq 0$.

# Edge-Edge-Distance (III)

**Problems:**

- $\lambda_1^*$ or $\lambda_2^*$ are outside the parameter interval $[0, 1]$:
  - Explanation: The edges represent line segments, not straight lines.
  - In this case we know that at least one of the closest points is an end point of an edge.
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- The denominator in equation 3 vanishes:
  - Explanation: The edges are parallel.
  - Also in this case we have that at least one of the closest points is an end point of an edge.
  - Solution: same as above.

# Vertex-Face-Distance

Let $\Sigma(f)$ denote the plane on which the face $f$ is embedded, i.e.

$$\Sigma(f) = \left\{ x \in \mathbb{R}^3 \,\middle|\, \mathbf{n}^T \mathbf{x} = n_0,\ \|\mathbf{n}\| = 1 \right\}.$$

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- $\overline{\mathbf{p}}$ is also the closest point of $f$ to $\mathbf{p}$, iff it is lying inside the polygon.
- Otherwise, a closest point of $f$ to $\mathbf{p}$ is located on one of the boundary edges of the polygon.

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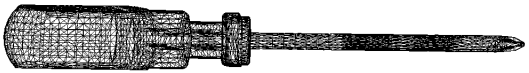

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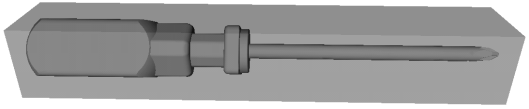

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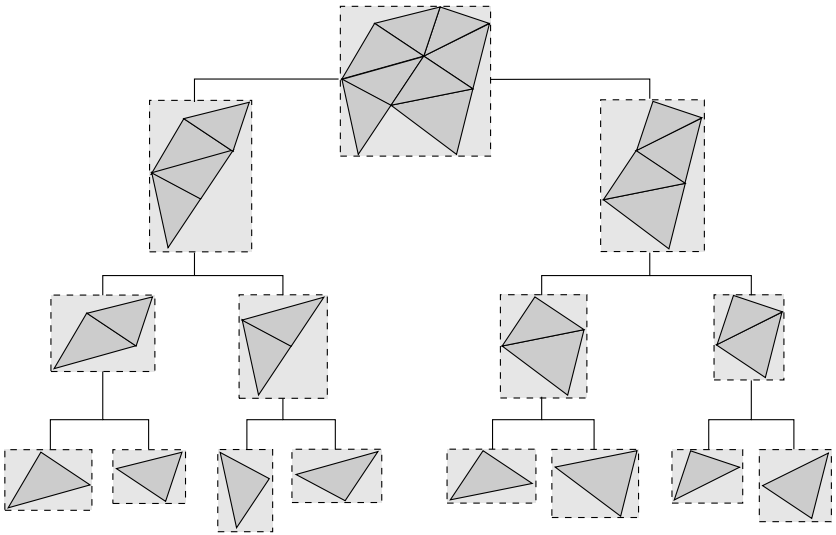
**Types of Bounding Volumes:**

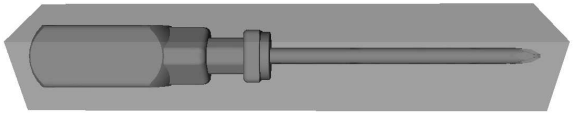
- Spheres [*Hubbard95*],[*Palmer,Grimsdale95*]
- Oriented Bounded Boxes (OBB) [*Gottschalk,Lin,Manocha96*]
- Fixed-Directions Hulls ($FDH_k$ or k-DOPS) [*Held,Klosowski,Mitchell96*]
  Special case: Axis-Aligned Bounding Boxes (AABB) [*Zachmann,Felger95*]
- Swept Sphere Volumes [*Larsen,Gottschalk,Lin,Manocha99*]

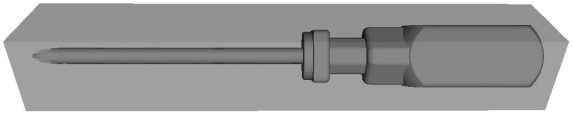
# Bounding Volume Hierarchy

**Definition 3 (Bounding Volume Hierarchy)**
*A bounding volume hierarchy is a tree structure successively refining the boundary of a polyhedron (or even a polygon soup) and covering each refinement by a set of bounding volumes.*



# Distance Computation Using BV-Hierarchies





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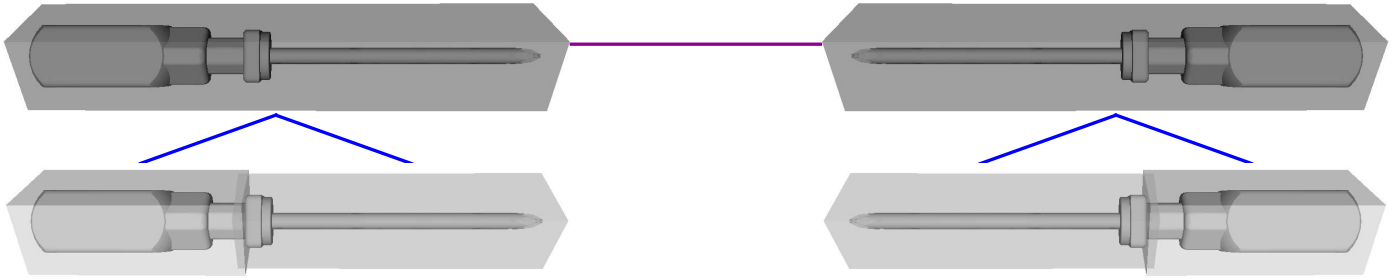

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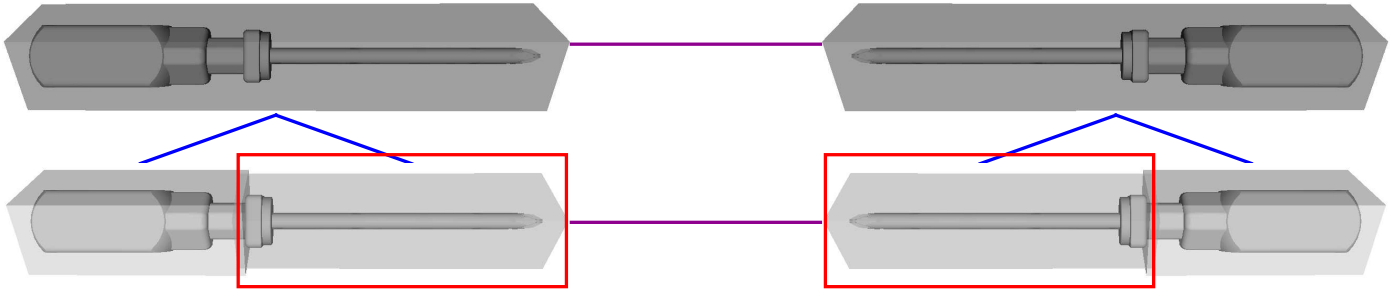

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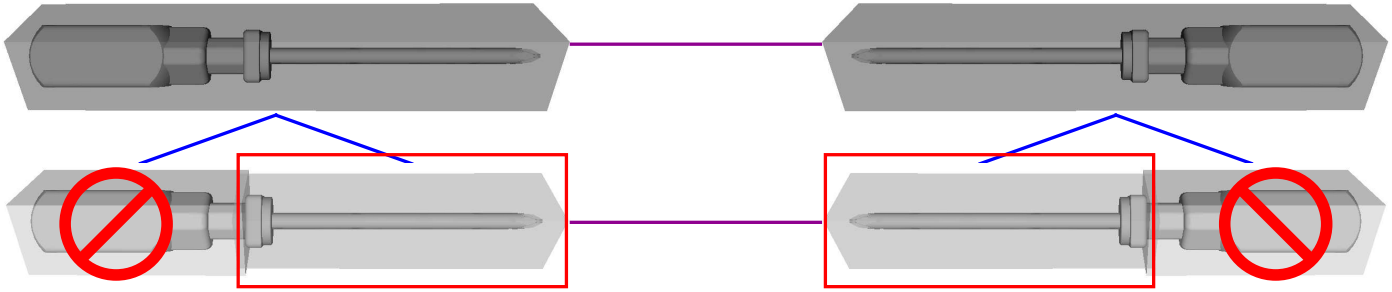

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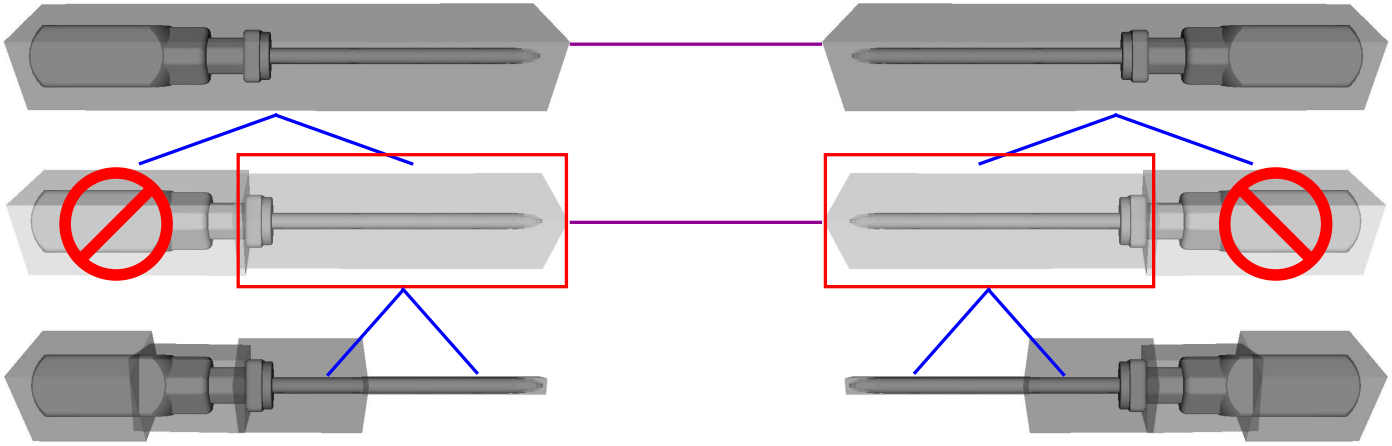

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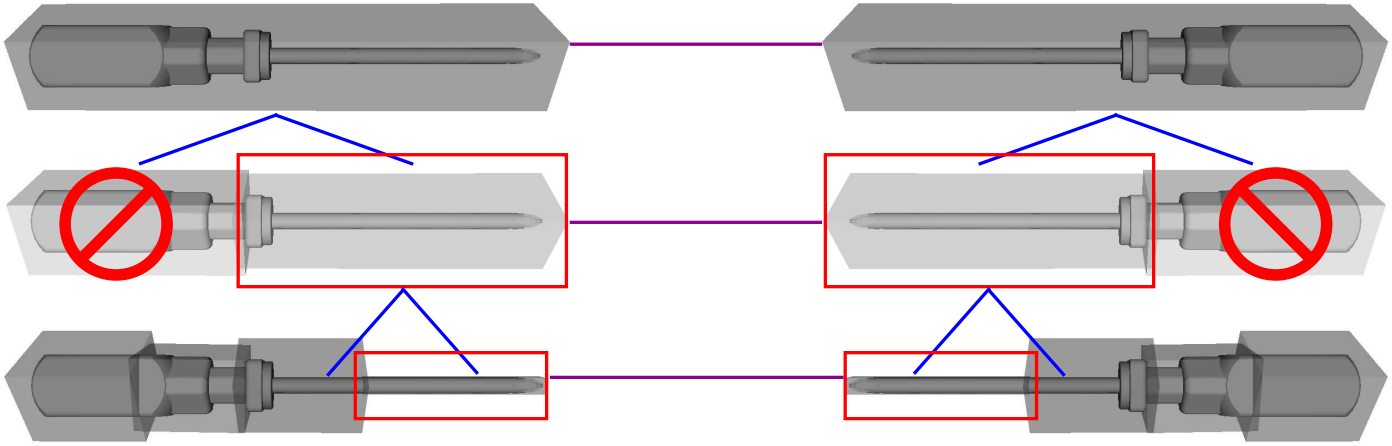

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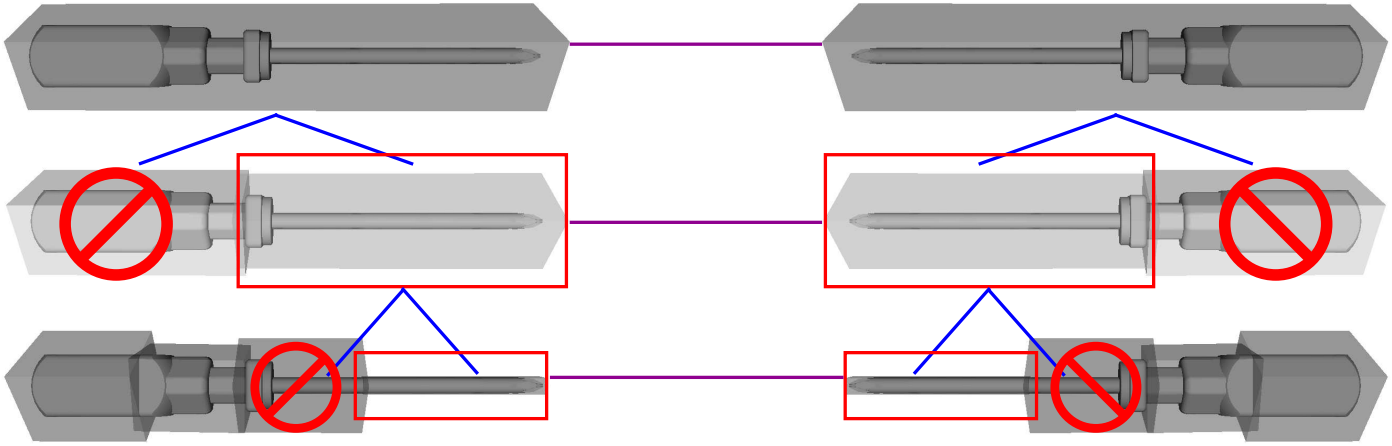

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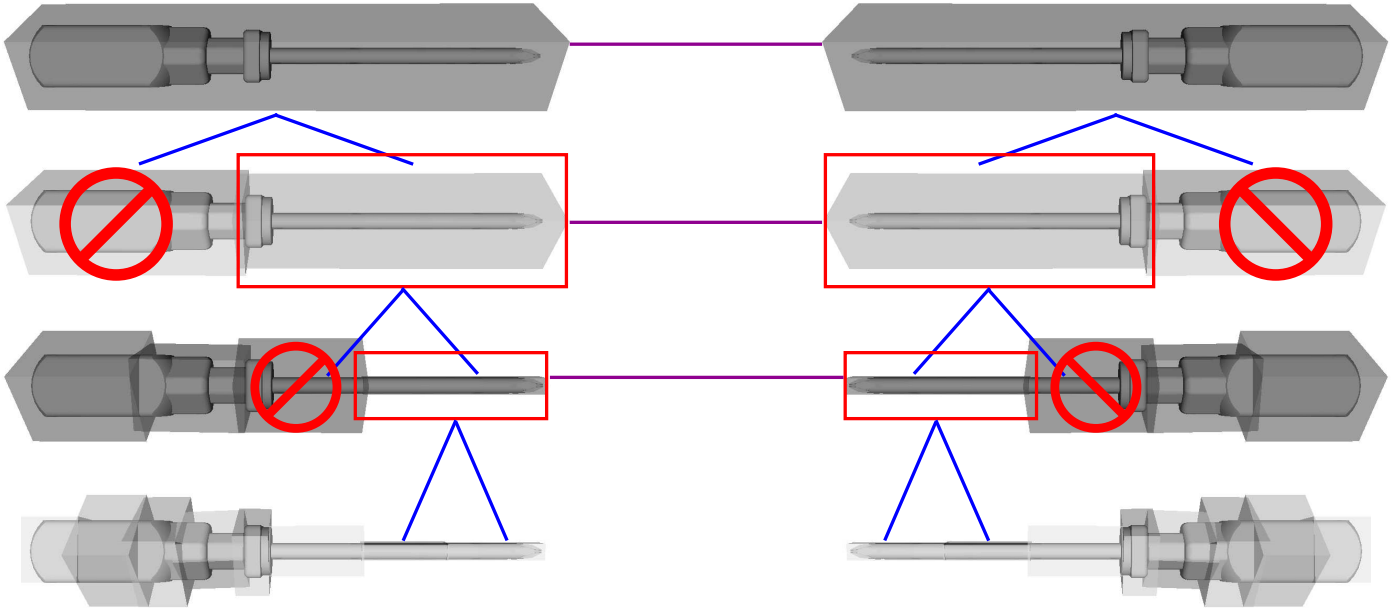

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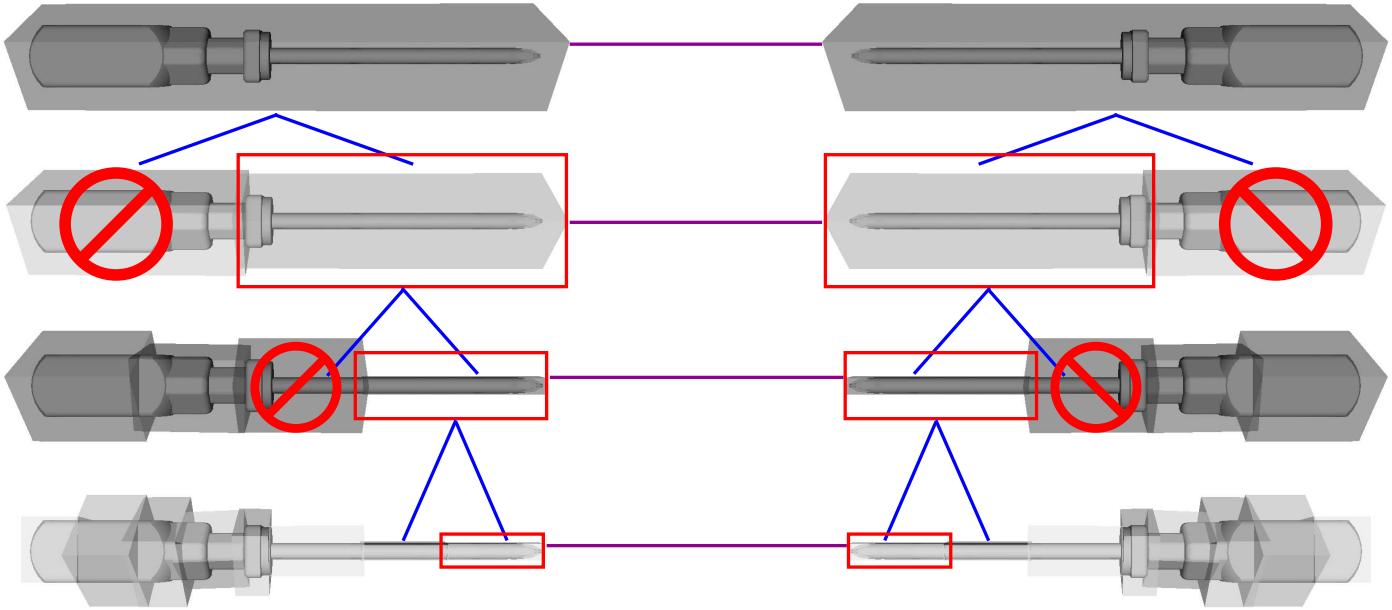

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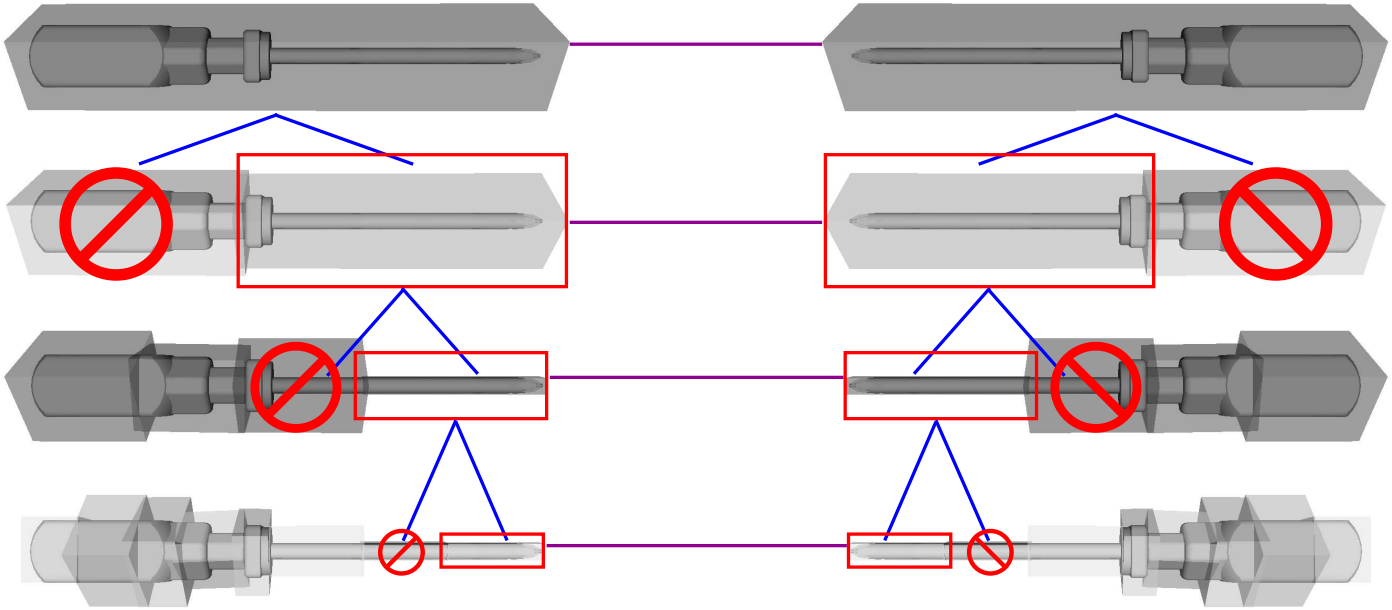

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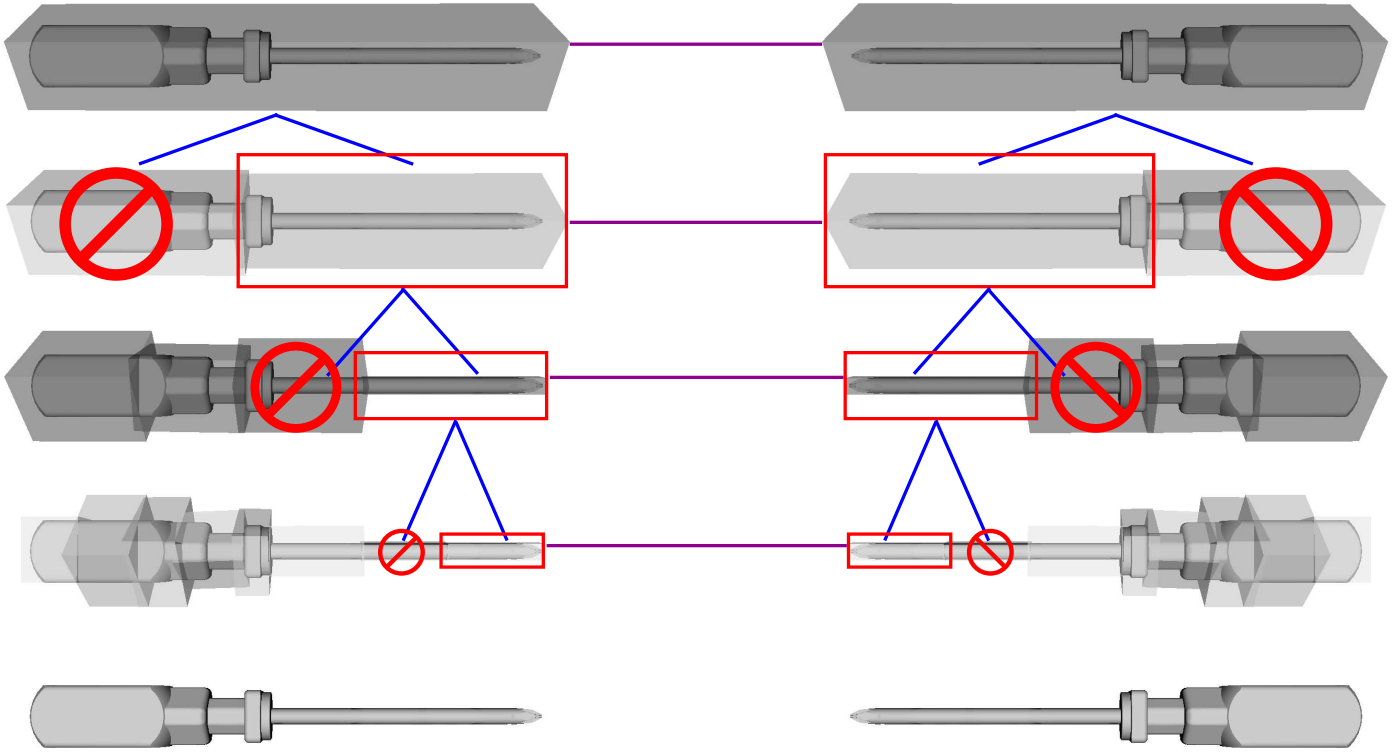

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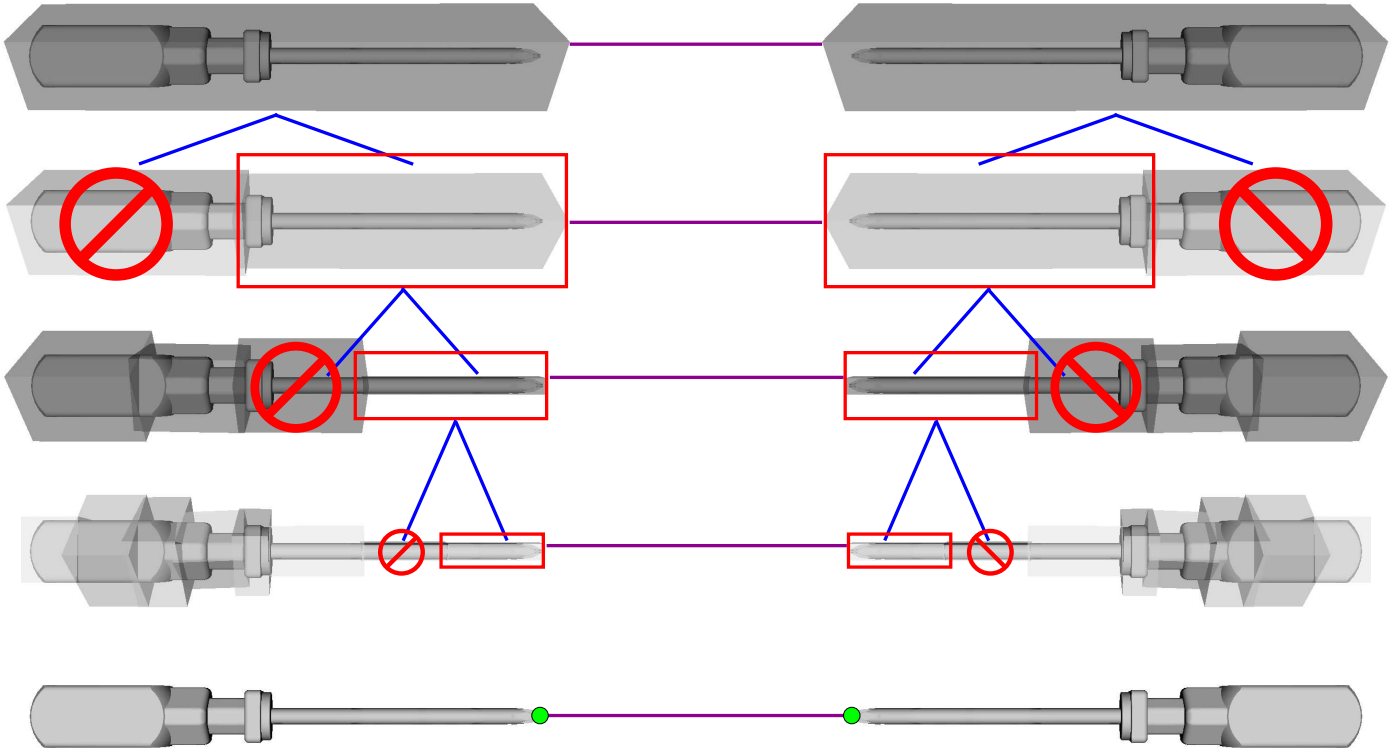

# Implementing the Algorithm

**Topics to discuss:**

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- To build the hierarchy we need
  - a strategy for partitioning a given face set,
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- To prune subtrees in the hierarchy we need
  - efficient algorithms to update bounding volumes after object movement,
  - efficient algorithms to compute the distance between bounding volumes.

# Minimal Bounding Volumes

**Task:**

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  **Definition 4 (Directed HAUSDORFF Distance)**
  *Given two point sets* $A$ *and* $B$*, then the value*

$$\delta_H(A, B) := \max_{a \in A} \min_{b \in B} \delta(\mathbf{a}, \mathbf{b})$$

  *is called directed* HAUSDORFF*-distance from* $A$ *to* $B$*.*

# Bounding Sphere of Minimal Volume

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- The minimization problem can be solved using methods of non-linear convex optimization.
- In practice one uses a combinatorial algorithm [*Welzl91*] that incrementally computes the minimal enclosing sphere in expected time $O(|\mathcal{V}|)$.

# OBB of Minimal Volume

The OBB is represented by a matrix $\mathbf{D} = [\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3]$ of normalized box directions, a center $\mathbf{c}$ and a vector of extends $\overline{\mathbf{d}}$. The volume is given by

$$V(B) = \prod_{i=1}^{3} \overline{d}_i \,,$$

$$\text{with} \quad \overline{d}_i = \Big( \max_{v \in \mathcal{V}(\mathcal{F})} \mathbf{d}_i^T \boldsymbol{v} - \min_{v \in \mathcal{V}(\mathcal{F})} \mathbf{d}_i^T \boldsymbol{v} \Big), \quad i = 1, 2, 3 \,.$$

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Now we have to solve a non-linear and non-convex optimization problem:

$$\begin{aligned} \min \quad & \prod_{i=1}^{3} \Big( \max_{v \in \mathcal{V}(\mathcal{F})} \mathbf{d}_i(\alpha, \beta, \gamma)^T \boldsymbol{v} - \min_{v \in \mathcal{V}(\mathcal{F})} \mathbf{d}_i(\alpha, \beta, \gamma)^T \boldsymbol{v} \Big) \\ \text{s.t.} \quad & \alpha, \beta, \gamma \in [0, 2\pi] \end{aligned}$$

# Combinatorial Algorithms

- The algorithm of O'ROURKE exploits the following necessary condition to find the enclosing box of minimal volume:

  *A box of minimal volume circumscribing a convex polyhedron must have at least two adjacent faces flush with the edges of the polyhedron.* [*O'Rourke85*]

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  - Determining the eigenvectors of the covariance matrix gives the principal axes.

# Partitioning of Face Sets

**Task:**

*Let $\alpha \geq 2$ denote the degree of the hierarchy and $\mathcal{F}$ the given face set. Determine a partition $\{\mathcal{F}_1, \ldots, \mathcal{F}_\alpha\}$ of $\mathcal{F}$ and bounding volumes $H_1, \ldots H_\alpha$, s.t. a given measure of quality is maximized.*

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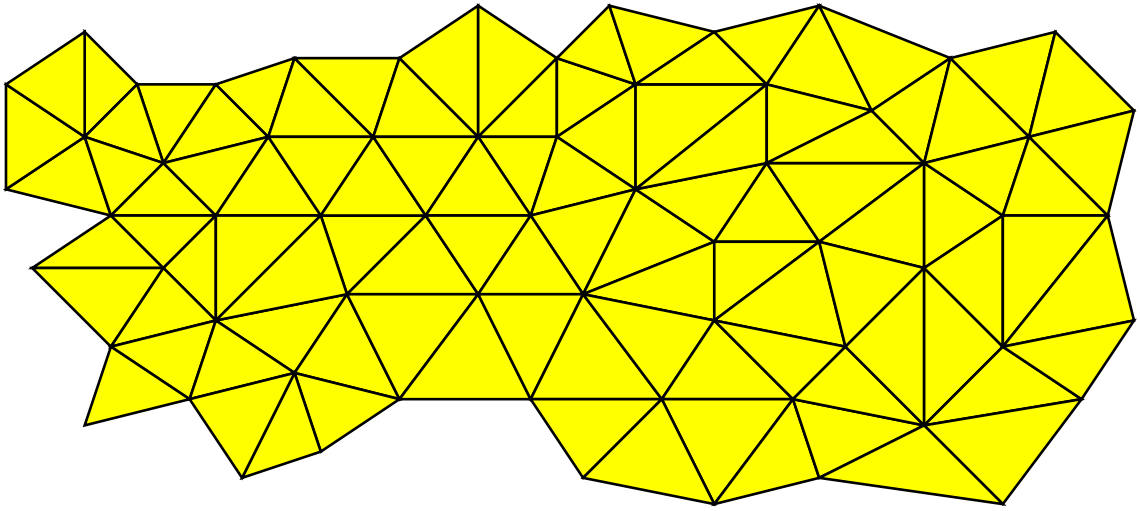
# Example of the Partitioning Heuristic



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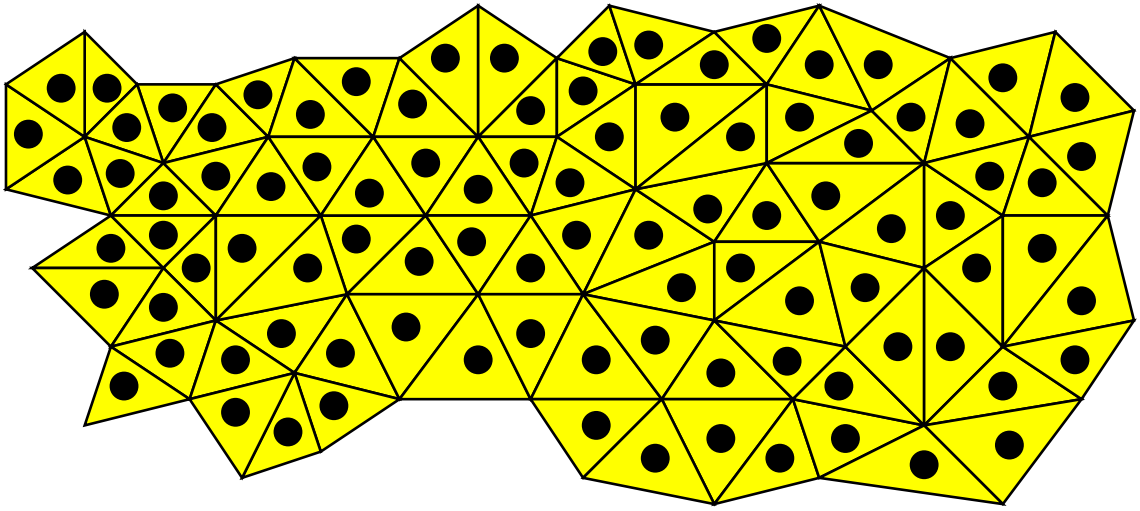

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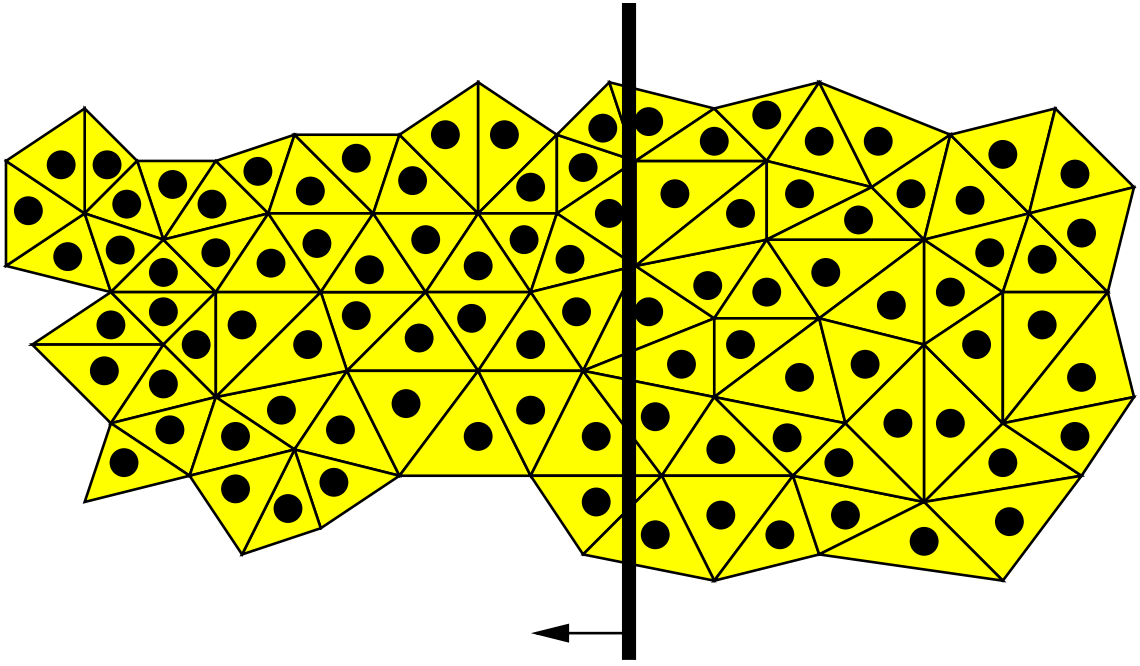

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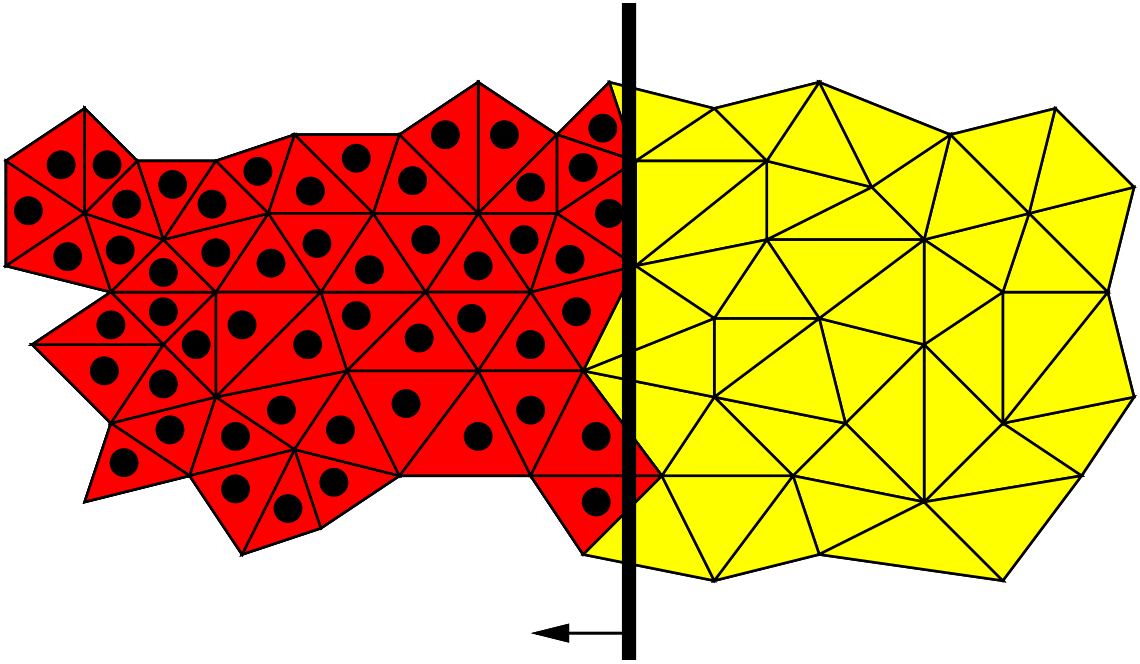

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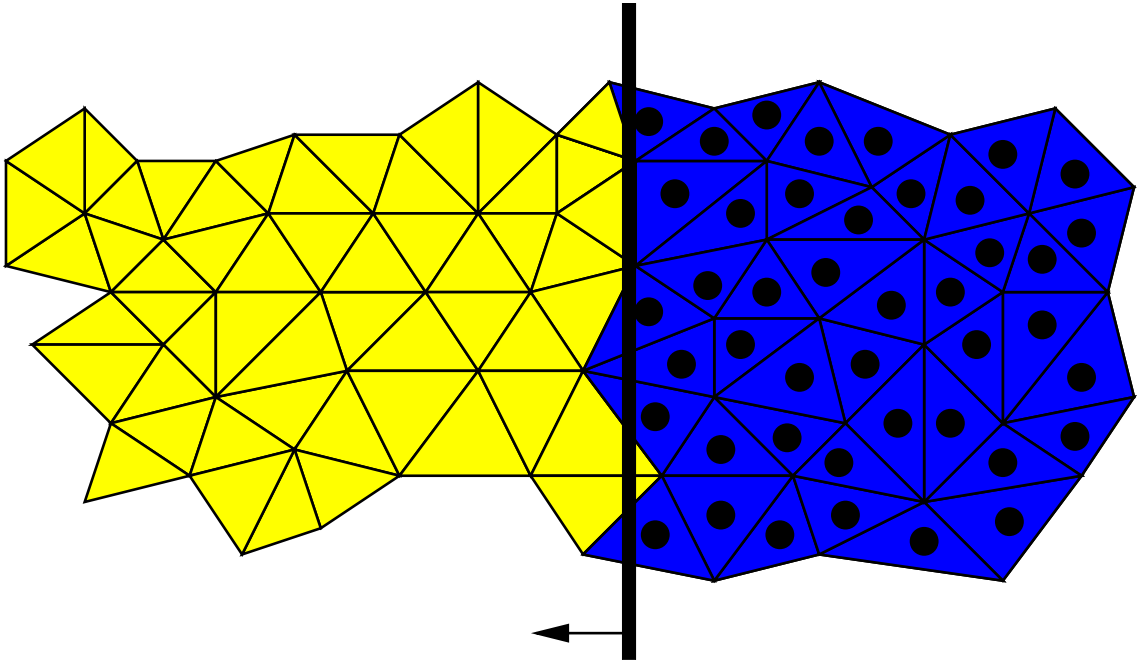

# Choosing the Direction of the Splitting Plane

The normal of the splitting plane is typically chosen as

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Thereby one can consider the following objective functions:

- Choose the axis, that minimizes the sum of the volumes or the maximal volume of the resulting child BVs:

$$\min_{d \in D} \sum_{j=1}^{\alpha} V\left(H_d\right) \quad \text{or} \quad \min_{d \in D} \max_{1 \le j \le \alpha} V\left(H_d\right) .$$

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- Choose the axis that yields the largest variance when projecting the reference points onto:

$$\max_{d \in D} \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} (\mathbf{d}^T \mathbf{p}_f - \mu)^2 \, , \qquad \mu := \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \mathbf{d}^T \mathbf{p}_f \, .$$

# Choosing the Position of the Splitting Plane

To choose the position of splitting plane one can consider the following objective functions:

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- Choose the point $\mathbf{p}_f$ that yields the median of the projected reference points.

# Distance Computation Between Bounding Volumes

**Tasks:**

- Compute the Euclidean distance between the geometric primitives: spheres, OBBs, AABBs, $FDH_k$...

- Minimize the effort spent on updating the bounding volumes during object movement.

# Updating Bounding Volumes

**Problem:**
The movement of objects in time implies that:

- the geometry has to be updated according to the new position and orientation of the object,
- the bounding volume hierarchy has to be updated, since, in general, the BVs are no longer optimal with respect the transformed face set.

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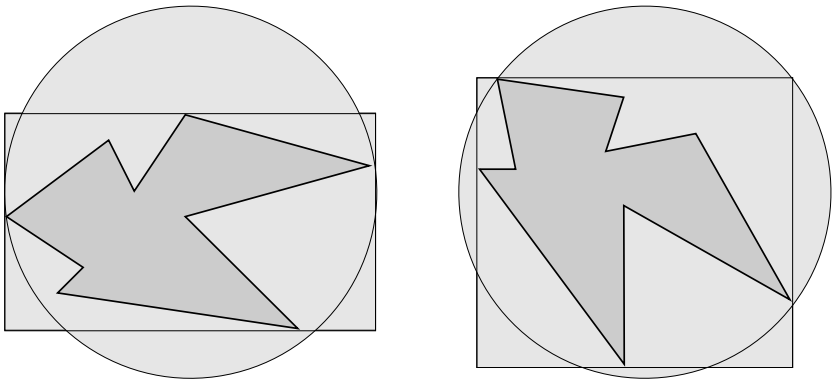

# Cost of BV-Update

The cost of updating bounding volumes depends on their closure properties under translation and rotation:

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**Reducing the update costs:**

- By transforming one object into the local coordinate system of the other, we only have to update the BV-hierarchy of one object.
- For $\mathrm{FDH}_k$s and AABBs, we can apply the object transformation to the BV of the previous time frame and compute the optimal BV of this transformed BV.

# Distance Computation Between OBBs

**The Edge Classification Algorithm [*Meyer86*]**

**Observation 2 (Box-Box-Distance)**
*If the boxes are disjoint, then the minimal distance is determined by*

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  - Assign the edges of one box to the VORONOI regions of the other box.
  - Use region specific proximity tests to compute the distance.
  - Break the edge into pieces if it passes through more than one cell.

# Distance Computation Between OBBs

**The Edge Classification Algorithm [*Meyer86*]**

**Observation 2 (Box-Box-Distance)**
*If the boxes are disjoint, then the minimal distance is determined by*

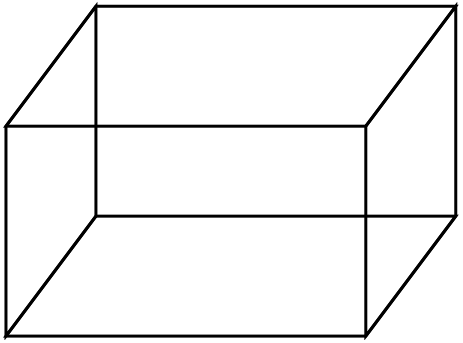
*(i) an edge of* $B_1$ *and the box* $B_2$ **or**

*(ii) a vertex of* $B_2$ *and a face of* $B_1$.

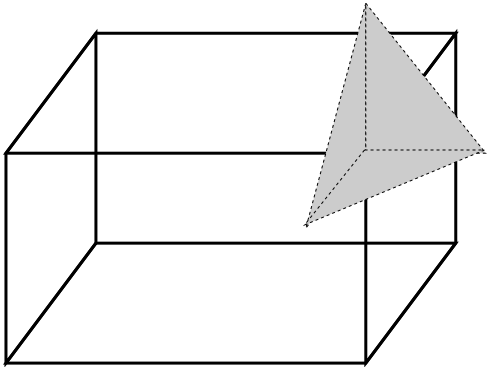
**Idea:**

- Case (ii): In the local coordinate system of $B_1$ one only needs coordinate comparisons.
- Case (i): Exploit box geometry to reduce the number of edge-edge tests:
  - Assign the edges of one box to the VORONOI regions of the other box.
  - Use region specific proximity tests to compute the distance.
  - Break the edge into pieces if it passes through more than one cell.

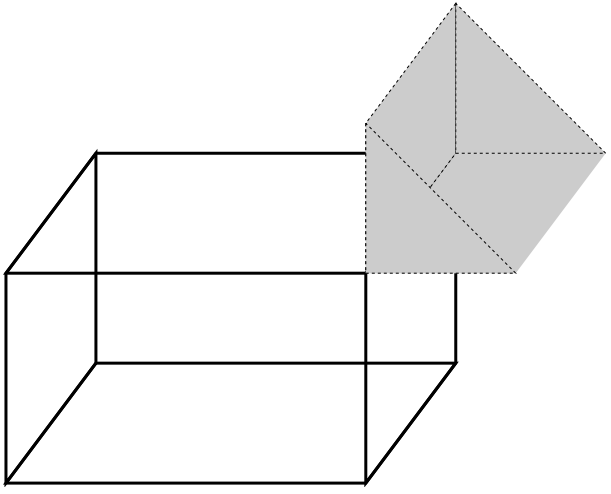
# Classification of Edges



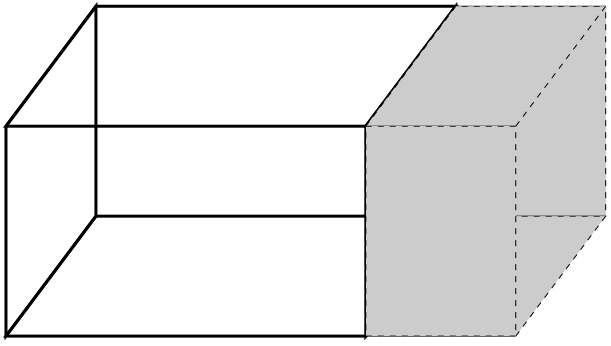
# Classification of Edges



# Classification of Edges



# Classification of Edges



# Classification of Edges